LINE PERCOLATION

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Abstract. We study a geometric bootstrap percolation model, *line percolation*, on the d-dimensional grid $[n]^d$. In line percolation with infection parameter r, infection spreads from a subset $A \subset [n]^d$ of initially infected lattice points as follows: if there is an axis parallel line $\mathcal{L}$ with r or more infected lattice points on it, then every lattice point of $[n]^d$ on $\mathcal{L}$ gets infected and we repeat this until the infection can no longer spread. The elements of the set A are usually chosen independently, with some density p, and the main question is to determine $p_c(n, r, d)$, the density at which percolation (infection of the entire grid) becomes likely. In this paper, we determine $p_c(n, r, 2)$ up to a factor of $1+o(1)$ and $p_c(n, r, 3)$ up to multiplicative constants as $n \to \infty$ for every fixed $r \in \mathbb{N}$. We also determine the size of the minimal percolating sets in all dimensions and for all values of the infection parameter.

1. Introduction

Bootstrap percolation models and arguments have been used to study a range of phenomena in various areas, ranging from crack formation, clustering phenomena, the dynamics of glasses and sandpiles to neural nets and economics; see [18, 3, 11] for a small sample of such applications. In this paper, we shall study a new geometric bootstrap percolation model defined on the d-dimensional grid $[n]^d$ with infection parameter $r \in \mathbb{N}$ which we call *r-neighbour line percolation*. Given $v \in [n]^d$, write $L(v)$ for the set of d axis parallel lines through v and let

$$L([n]^d) = \bigcup_{v \in [n]^d} L(v)$$

be the set of all axis parallel lines that pass through the lattice points of $[n]^d$. In line percolation, infection spreads from a subset $A \subset [n]^d$ of initially infected lattice points as follows: if there is a line $\mathcal{L} \in L\left([n]^d\right)$ with r or more infected lattice points on it, then every lattice point of $[n]^d$ on $\mathcal{L}$ gets infected. In other words, we have a sequence $A = A^{(0)} \subset A^{(1)} \subset \dots A^{(m)} \subset \dots$ of subsets of $[n]^d$ such that

$$A^{(m+1)} = A^{(m)} \cup \left\{ v \in [n]^d : \exists \mathcal{L} \in L(v) \text{ such that } |\mathcal{L} \cap A^{(m)}| \geq r \right\}.$$

The *closure* of A is the set $[A] = \bigcup_m A^{(m)}$ of eventually infected points. We say that the process *terminates* when no more newly infected points are added, i.e., when $A^{(m)} = [A]$. If

Date: 12 December 2013.
2010 *Mathematics Subject Classification.* Primary 60K35; Secondary 60C05.

all the points of $[n]^d$ are infected when the process terminates, i.e., if $[A] = [n]^d$, then we say that A *percolates*.

The classical model of *r-neighbour bootstrap percolation on a graph* was introduced by Chalupa, Leath and Reich [9] in the context of disordered magnetic systems and has since been extensively studied not only by mathematicians but by physicists and sociologists as well; for a small sample of papers, see, for instance, [1, 12, 13, 20]. In this model, a vertex of the graph gets infected if it has at least r previously infected neighbours in the graph. The model is usually studied in the random setting, where the main question is to determine the critical threshold at which percolation occurs. If the elements of the initially infected set are chosen independently at random, each with probability p, then one aims to determine the value p_c at which percolation becomes likely. In this regard, the r-neighbour bootstrap percolation model on $[n]^d$, with edges induced by the integer lattice $\mathbb{Z}^d$, has been the subject of large body of work; see [16, 6, 5], and the references therein.

On account of its inherent geometric structure, it is possible to construct other interesting bootstrap percolation models on the d-dimensional grid. In the past, this has involved endowing the grid with a graph structure other than the one induced by the integer lattice (which, in other words, is a Cartesian product of paths). In this direction, Holroyd, Liggett and Romik [17] considered r-neighbour bootstrap percolation on $[n]^2$ where the neighbourhood of a lattice point v is taken to be a "cross" centred at v, consisting of $r-1$ points in each of the four axis directions. Sharp thresholds for a model with an anisotropic variant of these "cross" neighbourhoods were obtained recently by Duminil-Copin and van Enter [10]. Gravner, Hoffman, Pfeiffer and Sivakoff [14] studied the r-neighbour bootstrap percolation model on $[n]^d$ with the edges induced by the Hamming torus where $u, v \in [n]^d$ are adjacent if and only if $u-v$ has exactly one nonzero coordinate; the Hamming torus, in other words, is the Cartesian product of complete graphs, which is perhaps the second most natural graph structure on $[n]^d$ after the grid. They obtained bounds on the critical exponents (i.e., $\log_n(p_c)$) which are tight in the case $d = 2$ and for small values of the infection parameter when $d = 3$.

The line percolation model we consider is a natural variant of the bootstrap percolation model on the Hamming torus studied by Gravner, Hoffman, Pfeiffer and Sivakoff. However, we should note that while all the other models mentioned above are r-neighbour bootstrap percolation models on *some underlying graph*, the line percolation model *is not*. Morally, line percolation is better thought of as an instance of the very general *neighbourhood family percolation* model introduced by Bollobás, Smith and Uzzell [8]. In the neighbourhood family percolation model, one starts by specifying a homogeneous (possibly infinite) collection of subsets of the grid for each point of the grid; a point of the grid becomes infected if all the points of some set in the collection associated with the point are previously infected. In their paper, Bollobás, Smith and Uzzell prove a classification theorem for neighbourhood family models and show that every such model is of one of three types: *supercritical*, *critical* or

subcritical. We note that line percolation is a natural geometric example of a supercritical neighbourhood family process. (Bollobás, Smith and Uzzell proved general bounds for the critical probabilities of supercritical and critical models; the analysis of subcritical models is more delicate and was later carried out by Balister, Bollobás, Przykucki and Smith [4].)

2. Our results

In this note, our main aim is to investigate what happens in the line percolation model when the initial set $A = A_p \subset [n]^d$ of infected points is determined by randomly selecting points from $[n]^d$, each independently with probability p. It would be natural to determine the values of p for which percolation is likely to occur. Let $\theta_p(n,r,d)$ denote the probability that such a randomly chosen initial set A_p percolates. We define the *critical probability* $p_c(n,r,d)$ by setting

$$p_c(n,r,d) = \inf\{p : \theta_p(n,r,d) \geq 1/2\}.$$

The primary question of interest is to determine the asymptotic behaviour of $p_c(n,r,d)$ for every $d, r \in \mathbb{N}$ as $n \to \infty$. Note that when the infection parameter $r = 1$, a set A of initially infected lattice points percolates if and only if $|A| > 0$; so in this paper, we restrict our attention to $r \geq 2$. In two dimensions, we are able to estimate the probability of percolation $\theta_p(n,r,2)$ up to constant factors for all $0 \leq p \leq 1$. We also determine $p_c(n,r,2)$ up to a factor of $1+o(1)$ as $n \to \infty$.

Theorem 1. *Fix $r, s \in \mathbb{N}$, with $r \geq 2$ and $0 \leq s \leq r-1$. Then as $n \to \infty$,*

$$\theta_p(n,r,2) = \Theta\left(n^{2s+1}(np)^{r(2s+1)-s(s+1)}\right) \quad \text{when} \quad n^{-1-\frac{1}{r-s-1}} \ll p \ll n^{-1-\frac{1}{r-s}}. \tag{1}$$

Also, $\theta_p(n,r,2) = \Theta(1)$ when $p \gg n^{-1-\frac{1}{r}}$. Furthermore,

$$p_c(n,r,2) \sim \lambda n^{-1-\frac{1}{r}}$$

where λ is the unique positive real number satisfying $\exp(-2\lambda^r/r!) = 1/2$.

The techniques used to obtain the above formula for $\theta_p(n,r,2)$ allow us to prove the following result about the critical probability in three dimensions, which is the main result of this paper.

Theorem 2. *Fix $r \in \mathbb{N}$, with $r \geq 2$, and let $s = \lfloor \sqrt{r+1/4} - 1/2 \rfloor$. Then as $n \to \infty$,*

$$p_c(n,r,3) = \Theta\left(n^{-1-\frac{1}{r-\gamma}}\right)$$

where $\gamma = \frac{r+s(s+1)}{2(s+1)}$.

The nature of the threshold at the critical probability is also worth investigating. We say that the model exhibits a sharp threshold at $p_c = p_c(n,r,d)$ if for any fixed $\epsilon > 0$, we have $\theta_{(1+\epsilon)p_c}(n,r,d) = 1 - o(1)$ and $\theta_{(1-\epsilon)p_c}(n,r,d) = o(1)$. It is not difficult to see from our proofs

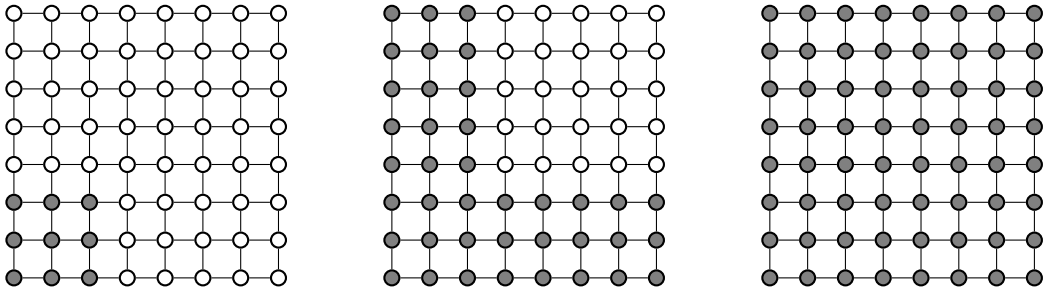

FIGURE 1. The spread of infection from $A = [3]^2$ in the 3-neighbour line percolation process on $[8]^2$.

of Theorems 1 and 2 that in stark contrast to the classical r-neighbour bootstrap percolation model on the grid, there is no sharp threshold at p_c when $d = 2, 3$. We expect similar behaviour in higher dimensions but we do not have a proof of such an assertion.

It is also an interesting question to determine the size of a minimal percolating set for r-neighbour line percolation on $[n]^d$ for any $d, r \in \mathbb{N}$ and $n \geq r$. It is easy to check that the set $[r]^d$ percolates (see Figure 1). We shall demonstrate that this is in fact optimal.

Theorem 3. *Let $d, r, n \in \mathbb{N}$, with $n \geq r$. Then the minimum size of a percolating set in the r-neighbour line percolation process on $[n]^d$ is r^d.*

Establishing this fact is much harder than it appears at first glance. The result is trivial when $d = 1$. When $d = 2$, it is not hard to demonstrate that any percolating set has size at least r^2. Consider a generalised two-dimensional line percolation model on $[n]^2$ where the infection thresholds for horizontal and vertical lines are r_h and r_v respectively; indeed, we recover the r-neighbour line percolation model when $r_h = r_v = r$. Let $M(r_h, r_v)$ denote the size of a minimal percolating set in this generalised model. Consider the first line $\mathcal{L}$ to be infected: if $\mathcal{L}$ is horizontal, then $\mathcal{L}$ must contain r_h initially infected points and furthermore, if the set of initially infected points is a percolating set, then the set of initially infected points not on $\mathcal{L}$ must constitute a percolating set for the generalised process with infection parameters r_h and $r_v - 1$. An analogous statement holds if $\mathcal{L}$ is vertical. It follows that

$$M(r_h, r_v) \geq \min\left(r_v + M(r_h - 1, r_v), r_h + M(r_h, r_v - 1)\right).$$

We obtain by induction that $M(r_h, r_v) \geq r_h r_v$ which implies in particular that $M(r, r) \geq r^2$. The argument described above depends crucially on the fact that a line has codimension one in a two-dimensional space. The incidence geometry of a collection of lines in the plane is essentially straightforward; this is no longer the case in higher dimensions and we need more delicate arguments to prove Theorem 3.

This paper is organised as follows. We collect together some useful facts about binomial random variables in Section 3. We consider line percolation in two dimensions in Section 4, and prove Theorem 1. In Section 5, we turn to line percolation in three dimensions and prove Theorem 2, thus obtaining an estimate for the critical probability which is tight up to multiplicative constants. In Section 6, we determine the size of minimal percolating sets for r-neighbour line percolation on $[n]^d$ and prove Theorem 3. We conclude the paper in Section 7 with some discussion.

A word on asymptotic notation; in this paper, we shall think of the infection parameter r as being fixed and study the behaviour of the percolation probability θ_p and the critical probability p_c as $n \to \infty$. Given functions $f(n), g(n)$, we write $f = O(g)$, or equivalently, $f \ll g$, if $f(n) \leq Cg(n)$ for some absolute constant C and all sufficiently large n. Similarly, we write $f = \Omega(g)$, or equivalently, $f \gg g$, if $g = O(f)$. If $f/g \to 0$ as $n \to \infty$, we say that $f = o(g)$; also, we write $f = \omega(g)$ if $g = o(f)$. If $f \ll g$ and $g \ll f$, we say that $f = \Theta(g)$. Finally, we write $f \sim g$ if $f = (1 + o(1))g$. In what follows, the various constants suppressed by the asymptotic notation are allowed to depend on the fixed infection parameter r, but of course, not on n or p.

3. Binomial random variables

We shall need some standard facts about binomial random variables. We collect these here for the sake of convenience. As is usual, for a random variable with distribution $\mathrm{Bin}(N, p)$, we write $\mu\,(= Np)$ for its mean.

Claim 4. *Let X be a random variable with distribution $\mathrm{Bin}(N, p)$ where $p \leq 1/2$. Then for any $k \geq 1$,*

$$\exp(-2\mu)\,(\mu/k)^k \leq \mathbb{P}(X = k) \leq \exp(-\mu)\,(2e\mu/k)^k.$$

Also, $\exp(-2\mu) \leq \mathbb{P}(X = 0) \leq \exp(-\mu)$. □

We shall make use of the following standard concentration result which first appeared in a paper of Bernstein and was later rediscovered by Chernoff and Hoeffding; see [19] for example.

Claim 5. *Let X be a random variable with distribution $\mathrm{Bin}(N, p)$. Then for any $0 < \delta < 1$,*

$$\mathbb{P}(|X - \mu| > \delta\mu) \leq \exp\left(\frac{-\delta^2\mu}{3}\right).$$ □

Finally, we shall make use of the following, easy claim.

Claim 6. *Let X be a random variable with distribution $\mathrm{Bin}(N, p)$ and suppose $\mu \ll 1$ as $N \to \infty$. Then for any $k \geq 0$,*

$$\mathbb{P}(X \geq k) = \Theta\left(\mathbb{P}(X = k)\right).$$ □

4. Line percolation in two dimensions

The proof of the following proposition is essentially identical to the proof of Theorem 2.1 in [14]; we reproduce it here for completeness.

Proposition 7. *Fix $r \in \mathbb{N}$, with $r \geq 2$, and let $\alpha > 0$ be a positive constant. If $p = \alpha n^{-1-\frac{1}{r}}$, then*

$$\theta_p(n, r, 2) \sim 1 - \exp\left(-2\alpha^r/r!\right).$$

Proof. The probability that a given line has $r+1$ or more initially infected points on it is bounded above by $\binom{n}{r+1}p^{r+1}$ which implies that the probability that any line has $r+1$ or more initially infected points on it is bounded above by $2n\binom{n}{r+1}p^{r+1} = O\left(n^{r+2}p^{r+1}\right) = O\left(n^{-1/r}\right)$. Consequently, asymptotically almost surely, no line has $r+1$ or more initially infected points on it.

Let E_h denote the event that some horizontal line contains r initially infected points and define E_v analogously. Clearly, the process terminates on the first step if neither E_h nor E_v hold; so $\theta_p \leq \mathbb{P}(E_h \cup E_v)$. Given a line $\mathcal{L}$, the probability that a particular line perpendicular to $\mathcal{L}$ has $r-1$ initially infected points (none of which are on $\mathcal{L}$) is $\Theta\left((np)^{r-1}\right) = \Theta\left(n^{-1+1/r}\right)$. Thus, the number of such lines is a binomial random variable with mean $\mu = \Omega\left(n^{1/r}\right)$. Since $\mu \to \infty$ as $n \to \infty$, by Claim 5, the probability that there exist at least r such lines is $1 - o(1)$. It follows that $\theta_p \sim \mathbb{P}(E_h \cup E_v)$.

The number of horizontal lines with r initially infected points is binomially distributed and it is easily seen to converge in distribution to a Poisson random variable with mean $(\alpha^r/r!)$. Thus $\mathbb{P}(E_h) \sim 1 - \exp(-\alpha^r/r!)$; similarly, $\mathbb{P}(E_v) \sim 1 - \exp(-\alpha^r/r!)$.

We now estimate $\mathbb{P}(E_h \cap E_v)$. Let $E_h \circ E_v$ denote the event that E_h and E_v occur disjointly. Now, E_h and E_v are increasing events, and so it follows from the FKG and BK inequalities that $\mathbb{P}(E_h \cap E_v) \geq \mathbb{P}(E_h)\mathbb{P}(E_v) \geq \mathbb{P}(E_h \circ E_v)$. Observe that $(E_h \cap E_v)\backslash(E_h \circ E_v)$ happens only if some lattice point v is initially infected and each of the two axis parallel lines through v contain $r-1$ initially infected points. It follows that

$$\mathbb{P}\left((E_h \cap E_v)\backslash(E_h \circ E_v)\right) = O\left(n^2 p(np)^{2r-2}\right) = O\left(n^{-1+1/r}\right)$$

and so $\mathbb{P}((E_h \cap E_v)\backslash(E_h \circ E_v)) = o(1)$. Consequently, we see that $\mathbb{P}(E_h \cap E_v) \sim \mathbb{P}(E_h)\mathbb{P}(E_v)$. Hence, we have $\mathbb{P}(E_h \cup E_v) \sim \mathbb{P}(E_h) + \mathbb{P}(E_v) - \mathbb{P}(E_h)\mathbb{P}(E_v)$ and the result follows. □

We shall now prove Theorem 1.

Proof of Theorem 1. It follows from Proposition 7 that $p_c(n, r, 2) \sim \lambda n^{-1-\frac{1}{r}}$ where λ is the unique positive real number satisfying $\exp\left(-2\lambda^r/r!\right) = 1/2$.

We now turn to estimating $\theta_p(n,r,2)$. To do so, we work with a modified two-dimensional line percolation process $A_p = G^{(0)} \subset G^{(1)} \subset \dots$ where

$$G^{(2m+1)} = G^{(2m)} \cup \left\{ v \in [n]^2 : |\mathcal{L} \cap G^{(2m)}| \geq r \text{ for some horizontal line } \mathcal{L} \in L(v) \right\}$$

and

$$G^{(2m+2)} = G^{(2m+1)} \cup \left\{ v \in [n]^2 : |\mathcal{L} \cap G^{(2m+1)}| \geq r \text{ for some vertical line } \mathcal{L} \in L(v) \right\}.$$

In other words, in going from $G^{(2m)}$ to $G^{(2m+1)}$, only horizontal lines are infected, and in going from $G^{(2m+1)}$ to $G^{(2m+2)}$, only vertical lines are infected, with the infection of lines happening as in the original line percolation process. Since $G^{(m)} \subset A^{(m)}$ and $A^{(m)} \subset G^{(2m)}$, percolation occurs in the original process if and only if it occurs in the modified process.

Note that A_p percolates if and only if some $G^{(m)}$ contains r or more parallel lines; indeed, in this case $G^{(m+1)} = [n]^2$. We stop the process as soon it produces r or more parallel fully infected lines (or reaches termination). Note that if percolation occurs, then it does so in at most $2r+1$ steps in the original process, and consequently, in at most $4r+2$ steps in the modified process.

Let h_i and v_i be the number of horizontal and vertical lines infected when going from $G^{(2i)}$ and $G^{(2i+1)}$ and from $G^{(2i+1)}$ to $G^{(2i+2)}$ respectively. The pair $(\mathbf{h} = \langle h_i \rangle, \mathbf{v} = \langle v_i \rangle)$ is called the *line-count* of the percolation process.

Given two sequences $\mathbf{h} = \langle h_i \rangle_{i=0}^{k}$ and $\mathbf{v} = \langle v_i \rangle_{i=0}^{k}$, we say that $(\mathbf{h}, \mathbf{v})$ is a *vertical line-count* if $(\mathbf{h}, \mathbf{v})$ is the line-count of a process which generates r fully infected vertical lines before it generates r fully infected horizontal lines, i.e., if

(1) $\sum_{i<k} v_i < r$,
(2) $\sum_{i \leq k} h_i < r$, and
(3) $\sum_{i \leq k} v_i \geq r$.

The definition of a *horizontal line-count* $(\mathbf{h} = \langle h_i \rangle_{i=0}^{k+1}, \mathbf{v} = \langle v_i \rangle_{i=0}^{k})$ is analogous.

Given a vertical line-count $\left(\mathbf{h} = \langle h_i \rangle_{i=0}^{k}, \mathbf{v} = \langle v_i \rangle_{i=0}^{k}\right)$, let us define its *(vertical) preface* to be the pair $(\mathbf{h}, \mathbf{v}')$ where $\mathbf{v}' = \langle v_i \rangle_{i=0}^{k-1}$. Similarly, the *(horizontal) preface* of a horizontal line-count $(\mathbf{h} = \langle h_i \rangle_{i=0}^{k+1}, \mathbf{v} = \langle v_i \rangle_{i=0}^{k})$ is the pair $(\mathbf{h}', \mathbf{v})$ where $\mathbf{h}' = \langle h_i \rangle_{i=0}^{k}$.

Given a vertical preface $(\mathbf{h}, \mathbf{v}')$, let $E_{(\mathbf{h},\mathbf{v}')}$ be the event that the process generates r fully infected vertical lines before it generates r fully infected horizontal lines and furthermore, the (vertical) line-count of the process has preface $(\mathbf{h}, \mathbf{v}')$. For a horizontal preface $(\mathbf{h}', \mathbf{v})$, define $E_{(\mathbf{h}',\mathbf{v})}$ analogously. We then note that

$$\theta_p(n,r,2) = \sum_{(\mathbf{h},\mathbf{v}')} \mathbb{P}\left(E_{(\mathbf{h},\mathbf{v}')}\right) + \sum_{(\mathbf{h}',\mathbf{v})} \mathbb{P}\left(E_{(\mathbf{h}',\mathbf{v})}\right)$$

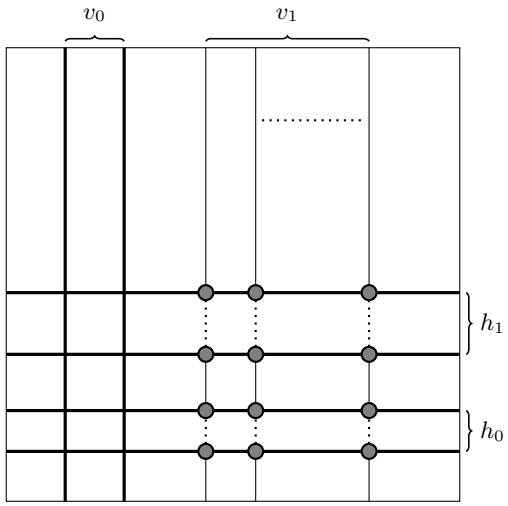

FIGURE 2. We need $(r - h_0 - h_1)$ initially infected points on some v_1 vertical lines to generate as many new fully infected vertical lines in the next step.

where the two sums are over all valid vertical and horizontal prefaces respectively. To specify a valid preface, we need to specify at most $2r$ distinct positive integers, each of which is at most r. So the number of valid prefaces is at most r^{2r}; consequently, to estimate the probability of percolation up to constant factors, it suffices to estimate the largest of the probabilities $\mathbb{P}\left(E_{(\mathbf{h},\mathbf{v}')}\right)$, $\mathbb{P}\left(E_{(\mathbf{h}',\mathbf{v})}\right)$.

Given $s \in \{0, 1, \ldots, r-1\}$, we see that $n\,(np)^{r-s} \ll 1$ and $n\,(np)^{r-s-1} \gg 1$ when $n^{-1-\frac{1}{r-s-1}} \ll p \ll n^{-1-\frac{1}{r-s}}$.

Let us say that a vertical preface $(\mathbf{h} = \langle h_i \rangle_{i=0}^{k}, \mathbf{v}' = \langle v_i \rangle_{i=0}^{k-1})$ is *slow* if $\sum_{i<k} v_i \leq s$ and $\sum_{i<k} h_i \leq s$. Similarly, let us say that a horizontal preface $\left(\mathbf{h}' = \langle h_i \rangle_{i=0}^{k}, \mathbf{v} = \langle v_i \rangle_{i=0}^{k}\right)$ is *slow* if $\sum_{i<k} v_i \leq s$ and $\sum_{i\leq k} h_i \leq s$.

The notion of a slow preface is motivated by the following observation. Suppose that at some stage in the process, we have l parallel fully infected lines where l is such that $n\,(np)^{r-l} \gg 1$. Then it follows from Claim 4 that with probability $\Omega\,(1)$, there exist r lines perpendicular to these l lines, each containing $r - l$ initially infected points none of which lie on lines infected earlier (of which there are at most $2r$). These r perpendicular lines become infected in the next step; consequently, we have percolation with probability $\Theta\,(1)$. It follows that given any (horizontal or vertical) preface $(\mathbf{x}, \mathbf{y})$, there exists a slow (horizontal or vertical) preface $(\mathbf{x}', \mathbf{y}')$, such that $\mathbb{P}\left(E_{(\mathbf{x},\mathbf{y})}\right) = O\left(\mathbb{P}\left(E_{(\mathbf{x}',\mathbf{y}')}\right)\right)$. Thus, to estimate θ_p, it suffices to restrict our attention to slow prefaces.

If at some stage in the process, we have l parallel fully infected lines where l is such that $n\,(np)^{r-l} \ll 1$, then the probability that the process generates exactly l' new fully infected

lines perpendicular to these l lines in the next step (see Figure 2) is easily seen to be

$$\Theta\left(\binom{n}{l'}\left((np)^{r-l}\right)^{l'}\left(1-(np)^{r-l}\right)^{n-l'}\right)=\Theta\left(\left(n\,(np)^{r-l}\right)^{l'}\right).$$

Given a slow, vertical preface $(\mathbf{h},\mathbf{v}')$, let us write $h=\sum_{i\leq k}h_i$. We consider two cases.

4.1. **Case 1:** $h\leq s$**.** If $h\leq s$, it follows from Claim 6 that

$$\mathbb{P}\left(v_k\geq r-\sum_{i<k}v_i\right)=\Theta\left(\mathbb{P}\left(v_k=r-\sum_{i<k}v_i\right)\right).$$

So in this case, we may assume that $\sum_{i\leq k}v_i=r$. We see that $\mathbb{P}\left(E_{(\mathbf{h},\mathbf{v}')}\right)$, up to constant factors, is given by,

$$(n(np)^r)^{h_0}\times(n(np)^{r-h_0})^{v_0}\times(n(np)^{r-v_0})^{h_1}\times\cdots\times(n(np)^{r-\sum_{i<k}v_i})^{h_k}\times(n(np)^{r-h})^{v_k}$$

which, on algebraic simplification, is seen to be $\Theta(n^{r+h}\,(np)^{r^2})$. This is maximised when $h=s$. Thus, in this case, we see that

$$\mathbb{P}\left(E_{(\mathbf{h},\mathbf{v}')}\right)=\Theta\left(n^{r+s}\,(np)^{r^2}\right).$$

4.2. **Case 2:** $h>s$**.** If $h>s$ on the other hand, we have $n\,(np)^{r-h}\gg 1$ and so the estimate for $\mathbb{P}\left(E_{(\mathbf{h},\mathbf{v}')}\right)$ becomes

$$(n(np)^r)^{h_0}\times(n(np)^{r-h_0})^{v_0}\times(n(np)^{r-v_0})^{h_1}\times\cdots\times(n(np)^{r-\sum_{i<k}v_i})^{h_k}\times 1$$

which in turn, on simplification, is seen to be $\Theta(n^{r+h}\,(np)^{r^2}\,(n\,(np)^{r-h})^{\sum_{i<k}v_i-r})$. Since $n\,(np)^{r-h}=\omega(1)$, the probability of $E_{(\mathbf{h},\mathbf{v}')}$ is maximised (disregarding constant factors) when $\sum_{i<k}v_i$ is maximal, subject to the condition that $\sum_{i<k}v_i\leq s$. Thus, we may assume that $\sum_{i<k}v_i=s$ and it follows that

$$\mathbb{P}\left(E_{(\mathbf{h},\mathbf{v}')}\right)=\Theta\left(n^{r+h}\,(np)^{r^2}\left(n\,(np)^{r-h}\right)^{s-r}\right)$$

which, on algebraic simplification, gives

$$\mathbb{P}\left(E_{(\mathbf{h},\mathbf{v}')}\right)=\Theta\left((n\,(np)^r)^s\left(n\,(np)^{r-s}\right)^h\right).$$

Since $n\,(np)^{r-s}\ll 1$, we may assume that $h=s+1$ and we conclude in this case that

$$\mathbb{P}\left(E_{(\mathbf{h},\mathbf{v}')}\right)=\Theta\left(n^{2s+1}\,(np)^{r(2s+1)-s(s+1)}\right).$$

We claim that the main contributions to θ_p come from Case 2. Note that

$$n^{2s+1}\,(np)^{r(2s+1)-s(s+1)}=n^{r+s}\,(np)^{r^2}\left(n\,(np)^{r-s}\right)^{s+1-r}\gg n^{r+s}\,(np)^{r^2}$$

because $\left(n\,(np)^{r-s}\right)^{s+1-r}\gg 1$; this is true since $\left(n\,(np)^{r-s}\right)\ll 1$ and $s+1-r\leq 0$.

Thus, we conclude that

$$\theta_p(n,r,2) = \Theta\left(n^{2s+1}(np)^{r(2s+1)-s(s+1)}\right) \quad \text{when } n^{-1-\frac{1}{r-s-1}} \ll p \ll n^{-1-\frac{1}{r-s}}$$

as required.

When $p \gg n^{-1-\frac{1}{r}}$, the probability that there exist r horizontal lines each containing r initially infected points is easily seen to be $\Omega(1)$. So $\theta_p(n,r,2) = \Theta(1)$ when $p \gg n^{-1-\frac{1}{r}}$. The result follows. □

5. The critical probability in three dimensions

We now turn our attention to the line percolation process in three dimensions. We shall now prove Theorem 2.

Proof of Theorem 2. We prove the upper and lower bounds separately. Let us start with the upper bound.

5.1. **Proof of the upper bound.** Unsurprisingly, it is easier to show that percolation occurs than to demonstrate otherwise. We start by bounding p_c from above. Let $p = Cn^{-1-\frac{1}{r-\gamma}}$ for some $C \gg 1$. Note that s, by definition, is the greatest natural number such that $s(s+1) \leq r$. Since $s(s+1) \leq r$ and $(s+1)(s+2) > r$, it is not hard to check that $\gamma = \frac{r+s(s+1)}{2(s+1)}$ satisfies

$$n^{-1-\frac{1}{r-s-1}} \ll n^{-1-\frac{1}{r-\gamma}} \ll n^{-1-\frac{1}{r-s}}, \tag{2}$$

and so it follows from (1) that

$$\theta_p(n,r,2) = \Theta\left(n^{2s+1}(np)^{r(2s+1)-s(s+1)}\right) = \Theta\left(C^{r(2s+1)-s(s+1)}n^{-1}\right).$$

We say that a plane $\mathcal{P}$ is internally spanned if $A_0 \cap \mathcal{P}$ percolates in the line percolation process restricted to $\mathcal{P}$. Choose any direction and consider the n (parallel) planes perpendicular to this direction. The number of such planes which are internally spanned is a binomial random variable with mean $\mu = \Omega\left(C^{r(2s+1)-s(s+1)}\right)$. Since $\mu \to \infty$ as $C \to \infty$, we see from Claim 5 that there exist r parallel internally spanned planes with probability at least $1/2$, provided C is a sufficiently large constant. So we have that $p_c(n,r,3) = O\left(n^{-1-\frac{1}{r-\gamma}}\right)$.

5.2. **Proof of the lower bound.** Next, suppose that $p = cn^{-1-\frac{1}{r-\gamma}}$ for some $c \ll 1$. We claim that the probability of percolation is at most $1/2$, provided c is a sufficiently small constant. We shall demonstrate this by proving something much stronger.

We shall track the number of planes with k parallel fully infected lines as the infection spreads for every $1 \leq k \leq s+1$ and show that these numbers are not too large when the process terminates with probability at least $1/2$.

We shall work with a modified three-dimensional line percolation process in which the infection spreads one line at a time. Let $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_{3n^2}$ be an ordering of the $3n^2$ lines of the three-dimensional grid. In this modified process, we have a sequence of subsets $A_p = H^{(0)} \subset H^{(1)} \subset \ldots H^{(m)} \subset \ldots$ of $[n]^3$ such that

$$H^{(m+1)} = \begin{cases} H^{(m)} \cup \mathcal{L}_k & \text{if } |\mathcal{L}_k \cap H^{(m)}| \geq r, \text{ where } k = m+1 \pmod{3n^2}, \\ H^{(m)} & \text{otherwise.} \end{cases}$$

Clearly, $H^{(m)} \subset A^{(m)} \subset H^{(3n^2 m)}$ and so A_p percolates in the original process if and only if it percolates in this modified process.

We run the modified three-dimensional process starting from A_p and if we find at stage m that

A: the number of planes containing k parallel fully infected lines will exceed $n^{1-\frac{k\gamma}{r-\gamma}}$ for some $1 \leq k \leq s+1$ at stage $m+1$, or

B: the process will terminate at stage $m+1$,

then we *stop the modified process at stage m*. Let E_A be the event that we the stop the modified process on account of Condition A.

Lemma 8. *In the modified process, we have*

$$\mathbb{P}(E_A) = O\left(\sum_{1 \leq k \leq s} c^{rk} + c^{r(2s+1)-s(s+1)}\right).$$

Proof. Let us write N_k for the number of planes containing k parallel fully infected lines when we stop the modified three-dimensional process. Since we are infecting lines one at a time, when we stop the process, we see that $N_k \leq n^{1-\frac{k\gamma}{r-\gamma}}$ for $1 \leq k \leq s$. Observe that $(s+1)\gamma/(r-\gamma) > 1$ since $(s+1)(s+2) > r$ and so $N_k = 0$ for $k \geq s+1$ since $n^{1-\frac{(s+1)\gamma}{r-\gamma}} < 1$. It follows that $N_0 = n - o(n)$.

We shall prove Lemma 8 by estimating the probability that a given plane contains k parallel fully infected lines when we stop the process. Let us fix a plane $\mathcal{P}$. Suppose that a point v of $\mathcal{P}$ gets infected before we stop the process and suppose further that v is not initially infected. Then v is either

(1) infected when a line perpendicular to $\mathcal{P}$ containing v has r other previously infected points on it (we call such points *boosted points*), or

(2) infected when a line in $\mathcal{P}$ containing v has r other previously infected points on it.

Let $A_{\mathcal{P}}$ denote the union of the boosted points and the initially infected points of $\mathcal{P}$. Observe that if we run the two-dimensional r-neighbour line percolation process on $\mathcal{P}$ starting from $A_{\mathcal{P}}$, we infect all the points of $\mathcal{P}$ that were infected in the modified three-dimensional process

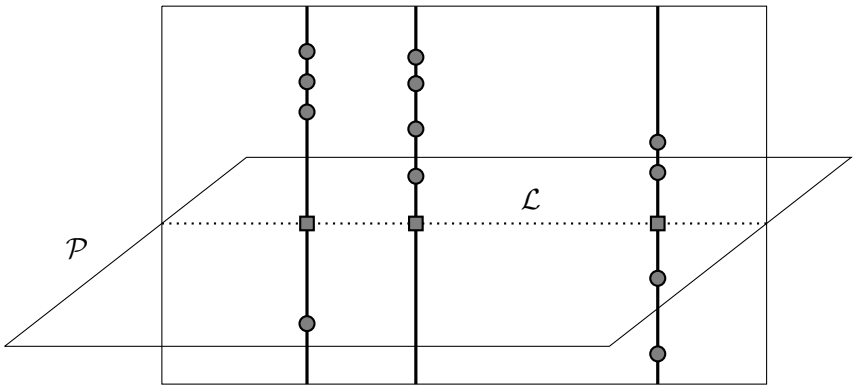

FIGURE 3. Boosted points on $\mathcal{L}$ in $\mathcal{P}$.

before it was stopped. Thus, the probability that $\mathcal{P}$ contains k parallel fully infected lines when we stop the modified three-dimensional process is bounded above by the probability that we generate k parallel fully infected lines in the two-dimensional r-neighbour line percolation process on $\mathcal{P}$ starting from $A_{\mathcal{P}}$.

Fix any arrangement of the boosted points in $\mathcal{P}$. Note that if we have k boosted points on a line $\mathcal{L}$ in $\mathcal{P}$, then this implies that the plane perpendicular to $\mathcal{P}$ which intersects $\mathcal{P}$ in $\mathcal{L}$ generated k parallel fully infected lines in the modified three-dimensional process before it was stopped (see Figure 3); consequently, the number of such lines $\mathcal{L}$ in $\mathcal{P}$ is at most N_k.

For $1 \le k \le s+1$, let E_k denote the event that the two-dimensional r-neighbour line percolation process on $\mathcal{P}$ starting from $A_{\mathcal{P}}$ generates k parallel fully infected lines.

Lemma 9. *Conditional on any arrangement in $\mathcal{P}$ of the boosted points, we have*

$$\mathbb{P}(E_k) = O\left(c^{rk} n^{-k\gamma/(r-\gamma)}\right)$$

for $1 \le k \le s$ and

$$\mathbb{P}(E_{s+1}) = O\left(c^{2(s+1)(r-\gamma)} n^{-1}\right).$$

Proof. As in the proof of Theorem 1, we consider the modified two-dimensional percolation process $A_{\mathcal{P}} = G^{(0)} \subset G^{(1)} \subset \dots$ on $\mathcal{P}$ where in going from $G^{(2m)}$ to $G^{(2m+1)}$, only horizontal lines are infected, and in going from $G^{(2m+1)}$ to $G^{(2m+2)}$, only vertical lines are infected. Let us stop this modified two-dimensional process on $\mathcal{P}$ as soon as it generates k or more parallel fully infected lines (or reaches termination).

For $0 \le j \le s$, let $h_{i,j}$ denote the number of horizontal lines containing j boosted points which are infected when going from $G^{(2i)}$ and $G^{(2i+1)}$. Let $v_{i,j}$ be defined analogously. We say that $(\mathbf{h} = \langle h_{i,j} \rangle, \mathbf{v} = \langle v_{i,j} \rangle)$ is the *full line-count* of the modified two-dimensional process; also, we define $h_i = \sum_j h_{i,j}$ and $v_i = \sum_j v_{i,j}$. Given $(\mathbf{h}, \mathbf{v})$, let $(\mathbf{h}^*, \mathbf{v}^*)$ be defined by setting $h^*_{i,0} = h_i$, $v^*_{i,0} = v_i$, and $h^*_{i,j} = v^*_{i,j} = 0$ for $1 \le j \le s$.

Let $E_{k,(\mathbf{h},\mathbf{v})}$ denote the event that the modified two-dimensional process on $\mathcal{P}$ generates k or more parallel fully infected lines and furthermore, the full line-count of the modified two-dimensional process on $\mathcal{P}$ is given by $(\mathbf{h},\mathbf{v})$.

For any $(\mathbf{h},\mathbf{v})$, we shall show that $\mathbb{P}\left(E_{k,(\mathbf{h},\mathbf{v})}\right) = O\left(\mathbb{P}\left(E_{k,(\mathbf{h}^*,\mathbf{v}^*)}\right)\right)$; in other words, we show that we may restrict our attention to the case where we never use any of the boosted points.

Having generated l parallel fully infected lines, let us consider the probability that the modified two-dimensional process generates exactly l' new fully infected lines perpendicular to these l lines in the next step. If $n\left(np\right)^{r-l} \gg 1$, then we see from Claim 4 that the probability of generating l' parallel fully infected lines in the next step where each of these l' new lines contain no boosted points is $\Omega\left(1\right)$. So suppose that $n\left(np\right)^{r-l} \ll 1$. In this case, the probability of generating l' parallel fully infected lines in the next step where each of these l' new lines contain no boosted points is

$$\Theta\left(\binom{N_0}{l'}\left(\left(np\right)^{r-l}\right)^{l'}\left(1-\left(np\right)^{r-l}\right)^{n-l'}\right) = \Theta\left(\left(n\left(np\right)^{r-l}\right)^{l'}\right)$$

since $N_0 = n - o\left(n\right)$. On the other hand, the probability of generating l' parallel fully infected lines in the next step where each of these l' new lines contain j boosted points for some $1 \leq j \leq s$ is

$$O\left(\binom{N_j}{l'}\left(\left(np\right)^{r-l-j}\right)^{l'}\right) = O\left(\left(n\left(np\right)^{r-l}\right)^{l'}\left(n^{1+\gamma/(r-\gamma)}p\right)^{-jl'}\right)$$

since $N_j < n^{1-j\gamma/(r-\gamma)}$. Observe that $n^{1+\gamma/(r-\gamma)}p = cn^{\frac{\gamma-1}{r-\gamma}}$, and since $\gamma = \frac{r+s(s+1)}{2(s+1)} \geq 1$ when $r \geq 2$, we see that $n^{1+\gamma/(r-\gamma)}p \gg 1$. It follows that for any $(\mathbf{h},\mathbf{v})$, we have

$$\mathbb{P}\left(E_{k,(\mathbf{h},\mathbf{v})}\right) = O\left(\mathbb{P}\left(E_{k,(\mathbf{h}^*,\mathbf{v}^*)}\right)\right).$$

Thus, to estimate $\mathbb{P}\left(E_k\right)$, we may restrict our attention to the events $E_{k,(\mathbf{h}^*,\mathbf{v}^*)}$. As in the proof of Theorem 1, we may suppose that $\sum_i v_i^* = k \leq s+1$ and that $\sum_i h_i^* = l < k$. Recall that s is the greatest natural number such that $s\left(s+1\right) \leq r$. Recall that $p = cn^{-1-\frac{1}{r-\gamma}}$, where $\gamma = \frac{r+s(s+1)}{2(s+1)}$, satisfies $n\left(np\right)^{r-s} \ll 1$ and $n\left(np\right)^{r-s-1} = \omega(1)$.

We shall mimick the proof of Theorem 1. Since $l \leq s$ and hence $n\left(np\right)^{r-l} \ll 1$, we see that the probability of $E_{k,(\mathbf{h}^*,\mathbf{v}^*)}$, up to constant factors, is given by

$$(n(np)^r)^{h_0^*} \times (n(np)^{r-h_0^*})^{v_0^*} \times (n(np)^{r-v_0^*})^{h_1^*} \times \cdots \times (n(np)^{r-\sum_{i<t} v_i^*})^{h_t^*} \times (n(np)^{r-\sum_{i\leq t} h_i^*})^{v_t^*}.$$

After some algebraic simplification, we see that

$$\mathbb{P}\left(E_{k,(\mathbf{h}^*,\mathbf{v}^*)}\right) = \Theta\left(n^{k+l}\left(np\right)^{rk+rl-kl}\right) = \Theta\left(n^k\left(np\right)^{rk}\left(n\left(np\right)^{r-k}\right)^l\right) \tag{3}$$

When $k \leq s$, we see that the estimate for the probability of $E_{k,(\mathbf{h}^*,\mathbf{v}^*)}$ in (3) is maximised by taking $l = 0$, from which we conclude that

$$\mathbb{P}(E_k) = O\left((n(np)^r)^k\right) = O\left(c^{rk} n^{-\frac{k\gamma}{r-\gamma}}\right).$$

On the other hand, when $k = s+1$, the estimate for the probability of $E_{k,(\mathbf{h}^*,\mathbf{v}^*)}$ in (3) is maximised by taking $l = s$, from which we conclude that

$$\mathbb{P}(E_{s+1}) = O\left(\left(n^{2s+1}(np)^{r(2s+1)-s(s+1)}\right)\right).$$

Using the fact that $\gamma = \frac{r+s(s+1)}{2(s+1)}$, we see that $\mathbb{P}(E_{s+1}) = O\left(c^{2(s+1)(r-\gamma)} n^{-1}\right)$ as required. This completes the proof of Lemma 9. □

Recall that E_A is the event that we stop modified three-dimensional process on account of the number of planes containing k parallel fully infected lines exceeding $\lfloor n^{1-\frac{k\gamma}{r-\gamma}} \rfloor$ for some $1 \leq k \leq s+1$.

From Lemma 9, we see that expected number of planes with k parallel fully infected lines when we stop the modified three-dimensional process is $O\left(c^{rk} n^{1-\frac{k\gamma}{r-\gamma}}\right)$ when $1 \leq k \leq s$ and $O\left(c^{r(2s+1)-s(s+1)}\right)$ when $k = s+1$. By Markov's inequality, the probability that the number of planes containing k parallel fully infected lines exceeds $\lfloor n^{1-\frac{k\gamma}{r-\gamma}} \rfloor$ is $O\left(c^{rk}\right)$ when $1 \leq k \leq s$ and $O\left(c^{r(2s+1)-s(s+1)}\right)$ when $k = s+1$ since $\lfloor n^{1-\frac{(s+1)\gamma}{r-\gamma}} \rfloor = 0$. Applying the union bound, we get

$$\mathbb{P}(E_A) = O\left(\sum_{1 \leq k \leq s} c^{rk} + c^{r(2s+1)-s(s+1)}\right).$$

This concludes the proof of Lemma 8. □

The required lower bound on p_c follows immediately from Lemma 8. The lemma implies that $\mathbb{P}(E_A) \to 0$ as $c \to 0$. Hence, for a suitably small constant c, the probability that the three-dimensional r-neighbour line percolation process with $p = cn^{-1-\frac{1}{r-\gamma}}$ generates a plane with $s+1$ parallel fully infected lines before reaching termination is less than $1/2$ since $n^{1-\frac{(s+1)\gamma}{r-\gamma}} < 1$. Consequently, the probability of percolation is also less than $1/2$. This implies that $p_c(n, r, 3) = \Omega\left(n^{-1-\frac{1}{r-\gamma}}\right)$ as required. This completes the proof of Theorem 2. □

6. Minimal percolating sets

In this section, we prove Theorem 3 which tells us the size of a minimal percolating set. We shall make use of the polynomial method which has had many unexpected applications in combinatorics; see [15] for a survey of many of these surprising applications. While linear algebraic techniques have previously been used to study bootstrap percolation processes (see [7]), we believe that this application of the polynomial method is new to the field.

Proof of Theorem 3. Suppose for the sake of contradiction that there is a set $A \subset [n]^d$ which percolates with $|A| < r^d$. We shall derive a contradiction using the polynomial method.

Proposition 10. *There exists a non-zero polynomial $P_A \in \mathbb{R}[x_1, x_2, \ldots, x_d]$ of degree at most $r-1$ in each variable which vanishes on A.*

Proof. Let $V \subset \mathbb{R}[x_1, x_2, \ldots, x_d]$ be the vector space of real polynomials in d variables of degree at most $r-1$ in each variable. The dimension of V is clearly r^d. Consider the evaluation map from V to $\mathbb{R}^{|A|}$ which sends a polynomial P to $(P(v))_{v \in A}$. Clearly, this map is linear. Since we assumed that $|A| < r^d$, this map has a non-trivial kernel. The existence of P_A follows. □

We shall use the polynomial P_A to follow the spread of infection. The following claim will yield a contradiction.

Proposition 11. *The polynomial P_A vanishes on $A^{(m)}$ for every $m \geq 0$.*

Proof. We proceed by induction on m. The claim is true when $m = 0$ since $A^{(0)} = A$. Now, assume P_A vanishes on $A^{(m)}$ and consider a line $\mathcal{L}$ which gets infected when going from $A^{(m)}$ to $A^{(m+1)}$. It must be the case that $|\mathcal{L} \cap A^{(m)}| \geq r$. Since P_A vanishes on $A^{(m)}$, the restriction of P_A to $\mathcal{L}$ disappears on $\mathcal{L} \cap A^{(m)}$. If the direction of $\mathcal{L}$ is $i \in [d]$, then the restriction of P_A to $\mathcal{L}$ is a univariate polynomial in the variable x_i of degree at most $r-1$. Since a non-zero univariate polynomial of degree at most $r-1$ has at most $r-1$ roots, the restriction of P_A to $\mathcal{L}$ has to be identically zero. Consequently, P_A vanishes on $A^{(m+1)}$. □

Since A percolates, we conclude that P_A vanishes on $[n]^d$. On the other hand, using the following proposition, the proof of which may be found in [2], we conclude that P_A cannot vanish on $[r+1]^d$.

Proposition 12. *Let $P = P(x_1, x_2, \ldots, x_d)$ be a polynomial in d variables over an arbitrary field F. Suppose that the degree of P as a polynomial in x_i is at most t_i for $1 \leq i \leq d$, and let $S_i \subset F$ be a set of at least $t_i + 1$ distinct elements of F. If $P(u_1, u_2, \ldots, u_d) = 0$ for every d-tuple $(u_1, u_2, \ldots, u_d) \in S_1 \times S_2 \times \cdots \times S_d$, then P is identically zero.* □

It follows from Proposition 12 that P_A is zero and we have a contradiction. This establishes Theorem 3. □

Remark. It follows from Theorem 3 that the size of a minimal percolating set in the r-neighbour bootstrap percolation model on $[n]^d$ with edges induced by the Hamming torus is at least $(r/d)^d$. On the other hand, it is possible to construct sets of size about $r^d/2d!$ which percolate. It would be interesting to determine the size of a minimal percolating set in this model exactly for all $d, r \in \mathbb{N}$; we suspect that the lower bound of $(r/d)^d$ is quite far from the truth.

7. Concluding remarks

There remain many challenging and attractive open problems, chief amongst which is the determination of $p_c(n, r, d)$ for all $d, r \in \mathbb{N}$. To determine $p_c(n, r, 3)$, we used a careful estimate for $\theta_p(n, r, 2)$ which is valid for all $0 \leq p \leq 1$. This estimate for $\theta_p(n, r, 2)$ depends crucially on the fact that the two-dimensional process reaches termination in a constant (depending on r, but not on n) number of steps. We believe that to determine $p_c(n, r, 4)$, one will need to determine $\theta_p(n, r, 3)$ for all $0 \leq p \leq 1$ but since it is not at all obvious that the three-dimensional process almost surely reaches termination in a constant number of steps, we suspect different methods will be necessary.

As remarked earlier, it is easily read out of our proofs that the line percolation model does not exhibit a sharp threshold at p_c in two or three dimensions. It would be interesting to prove an analogous statement for every $d, r \in \mathbb{N}$.

Acknowledgements

The first and second authors were partially supported by NSF grant DMS-1301614 and the second author also wishes to acknowledge support from EU MULTIPLEX grant 317532. Some of the research in this paper was carried out while the authors were visitors at Microsoft Research, Redmond. It was continued while the third and fourth authors were visitors at the University of Memphis. The authors are grateful to MSR, Redmond for their hospitality, and the third and fourth authors are additionally grateful for the hospitality of the University of Memphis. We would also like to thank Janko Gravner for bringing [14] to our attention.

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