Answer. The only such positive integers are $a = 2$ and $b = c = 1$.

Solution. For all positive integers n, let $(2+\sqrt{-3})^n = x_n + y_n\sqrt{-3}$, where x_n and y_n are integers. Then also $(2-\sqrt{-3})^n = x_n - y_n\sqrt{-3}$, and so

$$x_n^2 + 3y_n^2 = (x_n + y_n\sqrt{-3})(x_n - y_n\sqrt{-3}) = (2+\sqrt{-3})^n(2-\sqrt{-3})^n = 7^n.$$

We will show that, up to sign, x_n and y_n are the unique integers such that x_n and y_n are relatively prime and $x_n^2 + 3y_n^2 = 7^n$.

We proceed by induction on n. The base case $n = 1$ is clear. For the induction step, suppose that we have already established uniqueness for n, and that $x^2 + 3y^2 = 7^{n+1}$ with x and y relatively prime. Then

$$(x-2y)(x+2y) = x^2 - 4y^2 \equiv x^2 + 3y^2 \equiv 0 \pmod 7.$$

Without loss of generality, 7 divides $x - 2y$. (Otherwise, if 7 divides $x + 2y$, we replace y with $-y$.) Set $u = (2x+3y)/7$ and $v = (-x+2y)/7$. Then u and v are integers; $x + y\sqrt{-3} = (2+\sqrt{-3})(u + v\sqrt{-3})$; u and v are relatively prime; and $u^2 + 3v^2 = 7^n$. By the induction hypothesis, $u = \pm x_n$ and $v = \pm y_n$, and the rest is straightforward.

To solve the problem, we must find all n such that n divides y_n, and furthermore n and y_n/n are relatively prime. We proceed to show that the only such n is $n = 1$.

Let p be any prime such that p divides y_n for some n. We define the *order* of p to be the least positive integer d such that p divides y_d. We will show that p divides y_n if and only if d divides n.

To this end, observe that the sequence $y_1, y_2, \ldots$ satisfies the second-order homogeneous linear recurrence relation $y_n = 4y_{n-1} - 7y_{n-2}$. Consequently, modulo p this sequence is congruent to

$$\begin{array}{llllllll} y_1, & y_2, & \ldots, & y_{d-1}, & 0, & -7y_1y_{d-1}, & -7y_2y_{d-1}, & \ldots, & -7y_{d-1}^2, & 0, \\ 7^2y_1y_{d-1}^2, & 7^2y_2y_{d-1}^2, & \ldots, & 7^2y_{d-1}^3, & 0, & -7^3y_1y_{d-1}^3, & -7^3y_2y_{d-1}^3, & \ldots. \end{array}$$

The claim follows.

We proceed to show that, when $p \geq 5$, the order d of p divides either $p-1$ or $p+1$. To this end, it suffices to show that p divides either y_{p-1} or y_{p+1}. We work modulo p.

By expanding $(2+\sqrt{-3})^n$ we get that, for all n,

$$y_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} 2^{n-(2k+1)}(-3)^k.$$

Let r be the unique remainder modulo p such that p divides $4r + 3$. By Fermat's little theorem, p divides $r^{p-1} - 1 = (r^{\frac{p-1}{2}} - 1)(r^{\frac{p-1}{2}} + 1)$.

Suppose first that p divides $r^{\frac{p-1}{2}} - 1$. We will show that in this case p divides y_{p-1}.

Since, for all $\ell = 0, 1, \ldots, p-1$,

$$\binom{p-1}{\ell} = \frac{(p-1)!}{\ell!(p-1-\ell)!} \equiv \frac{1 \cdot 2 \cdot \cdots \cdot (p-1)}{1 \cdot 2 \cdot \cdots \cdot \ell \cdot (-[\ell+1]) \cdot (-[\ell+2]) \cdot \cdots \cdot (-[p-1])}$$
$$\equiv (-1)^{(p-1)-\ell} = (-1)^\ell,$$

we obtain

$$y_{p-1} = \sum_{k=0}^{\frac{p-3}{2}} \binom{p-1}{2k+1} 2^{(p-1)-(2k+1)}(-3)^k \equiv -2^{p-2} \sum_{k=0}^{\frac{p-3}{2}} r^k = -2^{p-2} \cdot \frac{r^{\frac{p-1}{2}} - 1}{r-1} \equiv 0,$$

as needed. (Here, $r \not\equiv 1 \bmod p$ because that would imply $p = 7$, whereas 7 does not divide y_n for any n.)

Otherwise, suppose that p divides $r^{\frac{p-1}{2}} + 1$. We will show that in this case p divides y_{p+1}.

Since, for all $\ell = 2, 3, \ldots, p-1$,

$$\binom{p+1}{\ell} = \frac{(p+1)!}{\ell!(p+1-\ell)!} \equiv 0,$$

we obtain

$$y_{p+1} = \sum_{k=0}^{\frac{p-1}{2}} \binom{p+1}{2k+1} 2^{(p+1)-(2k+1)}(-3)^k \equiv 2^p \left(\binom{p+1}{1} r^0 + \binom{p+1}{p} r^{\frac{p-1}{2}} \right) \equiv 2^p (r^{\frac{p-1}{2}} + 1) \equiv 0,$$

as needed.

Suppose that $n \geq 2$ and n divides y_n. We will show that either 2 or 3 divides n.

To this end, let p be the least prime factor of n and suppose, for the sake of contradiction, that $p \geq 5$. Let d be the order of p. Since p divides n and n divides y_n, we obtain that p divides y_n. Consequently, d divides n. However, d also divides at least one of $p-1$ and $p+1$. Therefore, all prime factors of d are smaller than p. Since $d \geq 2$ and d divides n, we arrive at a contradiction.

To complete the solution, we introduce the following lemma.

Lemma. Let k and ℓ be arbitrary positive integers. Then y_k divides $y_{k\ell}$ and $\gcd(y_k, y_{k\ell}/y_k) = \gcd(y_k, \ell)$.

Proof. Setting $\alpha = 2 + \sqrt{-3}$ and $\beta = 2 - \sqrt{-3}$, we obtain

$$y_{k\ell} = \frac{\alpha^{k\ell} - \beta^{k\ell}}{2\sqrt{-3}} = \frac{\alpha^k - \beta^k}{2\sqrt{-3}} \cdot \sum_{m=0}^{\ell-1} (\alpha^k)^m (\beta^k)^{(\ell-1)-m} = y_k \sum_{m=0}^{\ell-1} (x_k + y_k\sqrt{-3})^m (x_k - y_k\sqrt{-3})^{(\ell-1)-m}.$$

When we expand the above sum for $y_{k\ell}/y_k$, all terms that contain $\sqrt{-3}$ to an odd power cancel out and all of the remaining terms that contain y_k are congruent to zero modulo y_k. Gathering together the remaining terms, we arrive at

$$\frac{y_{k\ell}}{y_k} \equiv \sum_{m=0}^{\ell-1} x_k^m x_k^{(\ell-1)-m} = \ell \cdot x_k^{\ell-1} \pmod{y_k}.$$

Since x_k and y_k are relatively prime for all k, this completes the proof. □

The order of 2 is 2. Then, from $y_2 = 2^2$ and the Lemma, by induction on $\nu_2(n)$ we obtain that $\nu_2(y_n) = \nu_2(n) + 1$ for all even n. Similarly, the order of 3 is 3. Then, from $y_3 = 3^2$ and the Lemma, by induction on $\nu_3(n)$ we obtain that $\nu_3(y_n) = \nu_3(n) + 1$ for all n such that 3 divides n.

Suppose that $n \geq 2$ and n divides y_n. Then, as we established already, either 2 or 3 divides n. However, again as we established already, in the former case both n and y_n/n are multiples of 2, and in the latter case both n and y_n/n are multiples of 3. Therefore, n and y_n/n are not relatively prime.

This completes the solution.