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Problem. Triangle ABC has incenter I and excircles Ω_A, Ω_B, and Ω_C. Let ℓ_A be the line through the feet of the tangents from I to Ω_A, and define lines ℓ_B and ℓ_C similarly. Prove that the orthocenter of the triangle formed by lines ℓ_A, ℓ_B, and ℓ_C coincides with the Nagel point of triangle ABC.

(The Nagel point of triangle ABC is the intersection of segments AT_A, BT_B, and CT_C, where T_A is the tangency point of Ω_A with side BC, and points T_B and T_C are defined similarly.)

Solution. First we prove one lemma.

Lemma. Let point P lie outside of circles Ω_1 and Ω_2. Let ℓ_1 be the line through the feet of the tangents from P to Ω_1, define line ℓ_2 similarly, and let Q be the intersection point of lines ℓ_1 and ℓ_2. Then the midpoint of segment PQ lies on the radical axis of Ω_1 and Ω_2.

First proof. Let O_1 and O_2 be the centers of Ω_1 and Ω_2. Let S_1 and T_1 be the feet of the tangents from P to Ω_1, let M_1 be the midpoint of segment S_1T_1, and define points S_2, T_2, and M_2 similarly. We consider the case when points S_1 and S_2 lie on segments PT_1 and PT_2, respectively, and all other cases are analogous. Then

$$\begin{aligned}
\operatorname{power}(P,\Omega_1)-\operatorname{power}(P,\Omega_2) &= PS_1^2 - PS_2^2 \\
&= PM_1^2 + \tfrac{1}{4}S_1T_1^2 - PM_2^2 - \tfrac{1}{4}S_2T_2^2 \\
&= PQ^2 - QM_1^2 + \tfrac{1}{4}S_1T_1^2 - PQ^2 + QM_2^2 - \tfrac{1}{4}S_2T_2^2 \\
&= (QM_2 - M_2S_2)(QM_2 + M_2T_2) - \\
&\ (QM_1 - M_1S_1)(QM_1 + M_1T_1) \\
&= QS_2 \cdot QT_2 - QS_1 \cdot QT_1 \\
&= \operatorname{power}(Q,\Omega_2) - \operatorname{power}(Q,\Omega_1).
\end{aligned}$$

Observe that, when point X varies along line PQ, the difference $\operatorname{power}(X,\Omega_1)-\operatorname{power}(X,\Omega_2)$ depends linearly on X. Therefore, at the midpoint R of segment PQ we get that $\operatorname{power}(R,\Omega_1) = \operatorname{power}(R,\Omega_2)$, as needed. □

Second proof. (Ankan Bhattacharya) Let Γ be the circle on diameter PQ. Since the polar of P with respect to Ω_1 passes through Q, we get that Γ and Ω_1 are orthogonal. Similarly, Γ and Ω_2 are orthogonal as well. Therefore, the center of Γ lies on the radical axis of Ω_1 and Ω_2. On the other hand, the center of Γ is in fact the midpoint of segment PQ. □

Third proof. (Pavel Kozhevnikov) Let Ω be the circle with center P and zero radius. Denote the feet of the tangents from P to Ω_1 and Ω_2 by S_1, T_1, S_2, and T_2 as in the first proof. Then the midline m_1 of triangle PS_1T_1 opposite P is the radical axis of Ω and Ω_1. Similarly, the midline m_2 of triangle PS_2T_2 opposite P is the radical axis of Ω and Ω_2. Since the midpoint of segment PQ lies on both lines m_1 and m_2, we conclude that it is the radical center of Ω, Ω_1, and Ω_2. Thus it also lies on the radical axis of Ω_1 and Ω_2. □

Let Δ be the triangle formed by lines ℓ_A, ℓ_B, and ℓ_C. Let also H be the orthocenter of Δ. Observe that the sides of Δ are parallel to the exterior angle bisectors of triangle ABC. Thus the altitudes of Δ are parallel to the pairwise radical axes of Ω_A, Ω_B, and Ω_C. By the Lemma, it follows that the midpoint of segment IH is the radical center of Ω_A, Ω_B, and Ω_C.

On the other hand, let J be the incenter of the medial triangle of triangle ABC. Then it is well-known that J is the midpoint of segment IN.

We are only left to show that J is the radical center of Ω_A, Ω_B, and Ω_C. (In fact this is well-known, too, though not as widely as the theorem that J is the midpoint of segment IN.) Then it would follow immediately that points H and N coincide.

Here is one short proof. Let M be the midpoint of side BC, and let Ω_B and Ω_C touch line BC at points U and V, respectively. Then M is the midpoint of segment UV, too, and so M lies on the radical axis of Ω_B and Ω_C. Furthermore, since triangle ABC and its medial triangle are homothetic, it follows that lines MJ and AI are parallel, and so line MJ is perpendicular to the line through the centers of Ω_B and Ω_C. Thus line MJ coincides with the radical axis of Ω_B and Ω_C. Similarly, J lies on the radical axes of Ω_A with Ω_B and Ω_C as well. Therefore, J is indeed the radical center of Ω_A, Ω_B, and Ω_C. The solution is complete.

Remark. One more curious corollary of the Lemma is as follows. Let S be the intersection point of the perpendiculars to the sides of triangle ABC at their tangency points with the corresponding excircles. Let m_A be the line through the feet of the tangents from S to Ω_A, and define lines m_B and m_C similarly. Then the incenter of the triangle formed by lines m_A, m_B, and m_C coincides with the orthocenter of triangle ABC.

(Point S is known as the Bevan point of triangle ABC. It is also the circumcenter of the triangle whose vertices are the excenters of triangle ABC. For the proof of the above corollary, the important property of the Bevan point is that the circumcenter of triangle ABC coincides with the midpoint of segment IS, and so also J coincides with the midpoint of segment HS.)