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A SMALL CONVEX POLYTOPE WITH LONG EDGES, MANY VERTICES AND QUADRANGLE FACES ONLY

By

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It is known (see [1] or [2]) that there exists a 3-dimensional convex polytope of diameter 3 with arbitrary large number of vertices and edges of length at least 1. However, the constructions in these papers contain many triangle faces. We show a construction with quadrangle faces only by proving the following:

THEOREM 1. *There exists a 3-dimensional convex polytope in a sphere of diameter 3 with 2k+2 ($k \geq 4$) vertices, edge-lengths at least 1 and quadrangle faces only.*

To prove this we verify the following

LEMMA 1. *There exist a convex polygon with 2k ($k \geq 4$) vertices A_1, $A_2, \dots, A_{2k}$ and a point O in the interior of it, so that the intersection of OA_i and $A_{i-1}A_{i+1}$ is the midpoint of the former ($i = 1, 2, \dots, 2k$, $A_{2k+1} = A_1$).*

PROOF. Take an arbitrary "horizontal" line e and two points, A_1 and O on it, the former is on the "left". Let's take A_2 "below" e, so that $A_2A_1 = A_2O$. If we have already chosen A_i ($2 \leq i \leq 2k-5$), then denoting the midpoint of OA_i by F_i, we choose A_{i+1} on the line of $A_{i-1}F_i$, so that $A_{i-1}A_iA_{i+1}O$ is a convex quadrangle. We choose the A_i's for $1 < i \leq 2k-4$ in such a way that the orthogonal projection of A_i onto e is between A_1 and O. This is possible, since if this assumption holds for A_i, then it will hold for A_{i+1} as well, provided that A_{i+1} is close enough to F_i.

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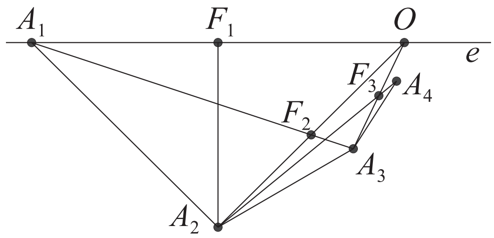
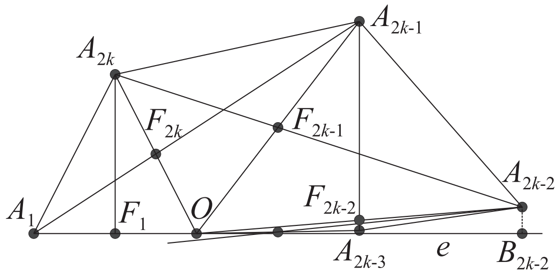


Now we choose A_{2k-3} on the line of $A_{2k-5}F_{2k-4}$, so that it is above the line e, and the distance between O and the orthogonal projection of A_{2k-3} onto e is not equal to $\frac{A_1O}{2}$ (this is clearly possible).

We choose A_{2k-2} on the line of $A_{2k-4}F_{2k-3}$, so that $A_{2k-3}F_{2k-2}$ is orthogonal to e (again this is possible). Let us denote the distance of A_{2k-2} from e by a_{2k-2}, the orthogonal projection of A_{2k-2} to e by B_{2k-2} and the ratio $\frac{OB_{2k-2}}{A_1O}$ by λ (> 0). We have $\lambda \neq 1$ by the construction of A_{2k-3}.

Finally we choose A_{2k-1} on the line of $A_{2k-3}F_{2k-2}$ "above" e, so that its distance from e is $\frac{(\lambda+2)^2}{2(\lambda-1)^2}a_{2k-2}$ and A_{2k} on the line of A_2F_1 "above" e, so that its distance from e is $\frac{3(\lambda+2)}{2(\lambda-1)^2}a_{2k-2}$. It is easy to check, that this is a proper polygon with the point O. ∎



Now the construction of the polytope is the following. Contract the polygon in lemma 1 from O so that the distances OA_i become smaller than $\sqrt{2}$. Take a parallel plane lying at distance 1 from the plane of the polygon and translate the polygon and the point O into that plane by a shift orthogonal to the planes. Denote the corresponding points by $A'_1, \dots A'_k, O'$. Choose two more points: O'', O''' so that O''', O, O', O'' are on a line in this order and $O'''O = OO' = O'O'' = 1$. Finally, let the vertices of the polytope be: $A_1, A_3, \dots, A_{2k-1}, A'_2, A'_4, \dots, A'_{2k}, O'', O'''$.

Because of the construction of the polygon the faces of this polytope are the quadrangles $O'''A_{i-1}A_{i+1}A'_i$ and $A_iA'_{i-1}A'_{i+1}O''$ (we have to check only that these fourtuples are coplanar and this is true, since the midpoint of $O'''A'_i$ is on $A_{i-1}A_{i+1}$). Thus the edges are the segments $O'''A_i$, $A_iA'_{i+1}$, A'_iA_{i+1}, A'_iO'', all of which are longer than 1 because $O'''A_i$ is longer than $O'''O = 1$, $A_iA'_{i+1}$ is longer, than $OO' = 1$, etc. Finally, the polytope is in the Thales sphere of $O'''O''$. This proves our theorem.

References

[1] K. BÖRÖCZKY, B. CSIKÓS, *Small convex polytopes with long edges and many vertices* (to appear in Discrete Comput. Geom.)

[2] K. BEZDEK, G. BLEKHERMAN, R. CONNELY and B. CSIKÓS, The polyhedral Tammes problem, *Arch. Math.* **76** (2001), 314–320.

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